Stern- und Planetenentstehung
Sommersemester 2020
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Lecture 3: Gas Flows and Turbulence

http://exp-astro.physik.uni-frankfurt.de/star_formation/index.php

VORLESUNG/LECTURE
Raum: Physik - 02.201a
dienstags, 12:00 - 14:00 Uhr

SPRECHSTUNDE:
Raum: GSC, 1/34, Tel.: 47433, (roellig@ph1.uni-koeln.de)
dienstags: 14:00-16:00 Uhr

Nr.	Thema	Termin
1	Observing the cold ISM	21.04.2020
2	Observing Young Stars	28.04.2020
3	Gas Flows and Turbulence Magnetic Fields and Magnetized Turbulence	05.05.2020
4	Gravitational Instability and Collapse	12.05.2020
5	Stellar Feedback	19.05.2020
6	Giant Molecular Clouds	26.05.2020
7	Star Formation Rate at Galactic Scales	02.06.2020
8	Stellar Clustering	09.06.2020
9	Initial Mass Function – Observations and Theory	16.06.2020
10	Massive Star Formation	23.06.2020
11	Protostellar disks and outflows – observations and theory	30.06.2020
12	Protostar Formation and Evolution	07.07.2020
13	Late Stage stars and disks – planet formation	14.07.2020

3 GAS FLOWS AND TURBULENCE

3.1 CHARACTERISTIC NUMBERS FOR FLUID FLOW

3.1.1 The Conservation Equations

Fluids are governed by a series of conservation laws

conservation of mass:

$$\frac{\partial}{\partial t}\rho = -\nabla \cdot (\rho\, \vec{v})$$

change of mass density at a fixed point is equal to minus the divergence of density times velocity = rate at which mass flows into or out of an infinitesimal volume around that point

conservation of momentum:

$$\frac{\partial}{\partial t}(\rho\, \vec{v}) = -\nabla \cdot (\rho\, \vec{v}\, \vec{v}) - \nabla P + \rho\, \nu\, \nabla^2\, \vec{v}$$

($\vec{v}\,\vec{v}$: tensor product, (outer product) -> index notation)

$$\frac{\partial}{\partial t}(\rho\, v_i) = -\frac{\partial}{\partial x_j} \cdot (\rho\, v_i v_j) - \frac{\partial}{\partial x_i} P + \rho\, \nu\, \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_j}\, v_i\right)$$

viscosity ν

$\rho\vec{v}$: density of momentum at a point

$\nabla \cdot (\rho\, \vec{v}\, \vec{v})$: analog to cons. of mass rate at which momentum is advected into or out of the point by the flow

∇P: rate at which pressure forces acting on the fluid changes its momentum

$\rho\, \nu\, \nabla^2\, \vec{v}$: rate at which viscosity redistributes momentum (This is the only term without analogous counterpart in Newton's second law for single particles)

ν: kinematic viscosity

The viscous term describes the change in fluid momentum due to diffusion of momentum from adjacent fluid elements.

fluid velocity= systematic flow + random velocity field

viscosity turns bulk motion into random motion => dissipation

3.1.2 The Reynolds Number and the Mach Number

To understand rel. importance of terms in momentum equation

⇨ dimensional analysis

system of characteristic size L and characteristic velocity V

(mol. cloud: $L{\sim}10\ \mathrm{pc}, V{\sim}5\ \mathrm{km\ \ s^{-1}}$)

the natural time scale for such a system: L/V

⇨ $\left(\frac{\partial X}{\partial t} \sim \frac{X}{\frac{L}{V}}\right)$ and $\left(\frac{\partial X}{\partial x} \sim \frac{X}{L}\right)$

momentum equation ($P = \rho c_s^2$, c_s=gas sound speed):

$$\frac{\rho V^2}{L} \sim \frac{\rho V^2}{L} + \frac{\rho c_s^2}{L} + \rho\nu\frac{V}{L^2} \qquad /: \frac{V^2}{L}\rho$$

$$1 \sim 1 + \underbrace{\frac{c_s^2}{V^2}}_{\frac{1}{M^2}} + \underbrace{\frac{\nu}{V\,L}}_{\frac{1}{Re}}$$

We define:

Mach number $\mathcal{M} \sim \frac{V}{c_s}$

Reynolds number $Re \sim \frac{L\,V}{\nu}$

If $\mathcal{M} \ll 1 \; then \, \frac{c_s^2}{V^2} \gg 1$ pressure term is important

$\mathcal{M} \gg 1$ pressure term unimportant

$\boxed{c_s = \sqrt{\frac{k_B T}{\mu m_H}}} = 0.18 \left(\frac{T}{10K}\right)^{1/2}, \quad \mu = 2.33$ (mean mass per particle assuming 1He per 10 H => 14/6)

⇨ $\frac{\mathcal{M} V}{c_s} \sim 20$ => pressure forces unimportant in mol. clouds

If $Re \lesssim 1$ viscous forces are important

$Re \gg 1$ viscous forces are unimportant

Reynolds number describes characteristic length scale $L \sim \nu / V$ in the flow

This is the length scale on which diffusion causes to flow to dissipate energy => large scale motion effectively dissipationless.

ideal gas: $\nu = 2\bar{u}\lambda$ $\bar{u}$: RMS molecular speed (order of c_s)

λ: particle mean free path
(inverse of cross-section times density)

$\lambda \sim \frac{1}{\sigma n} \sim [1(nm)^2(100cm^{-3})]^{-1} \sim 10^{12}$ cm

$\nu \sim 10^{16} \mathrm{cm}^2\,\mathrm{s}^{-1}$ and $Re \sim 10^9$

⇨ viscous forces are unimportant in molecular clouds

⇨ molecular clouds must be highly turbulent! (turbulent if $Re \gtrsim 10^3 - 10^4$)

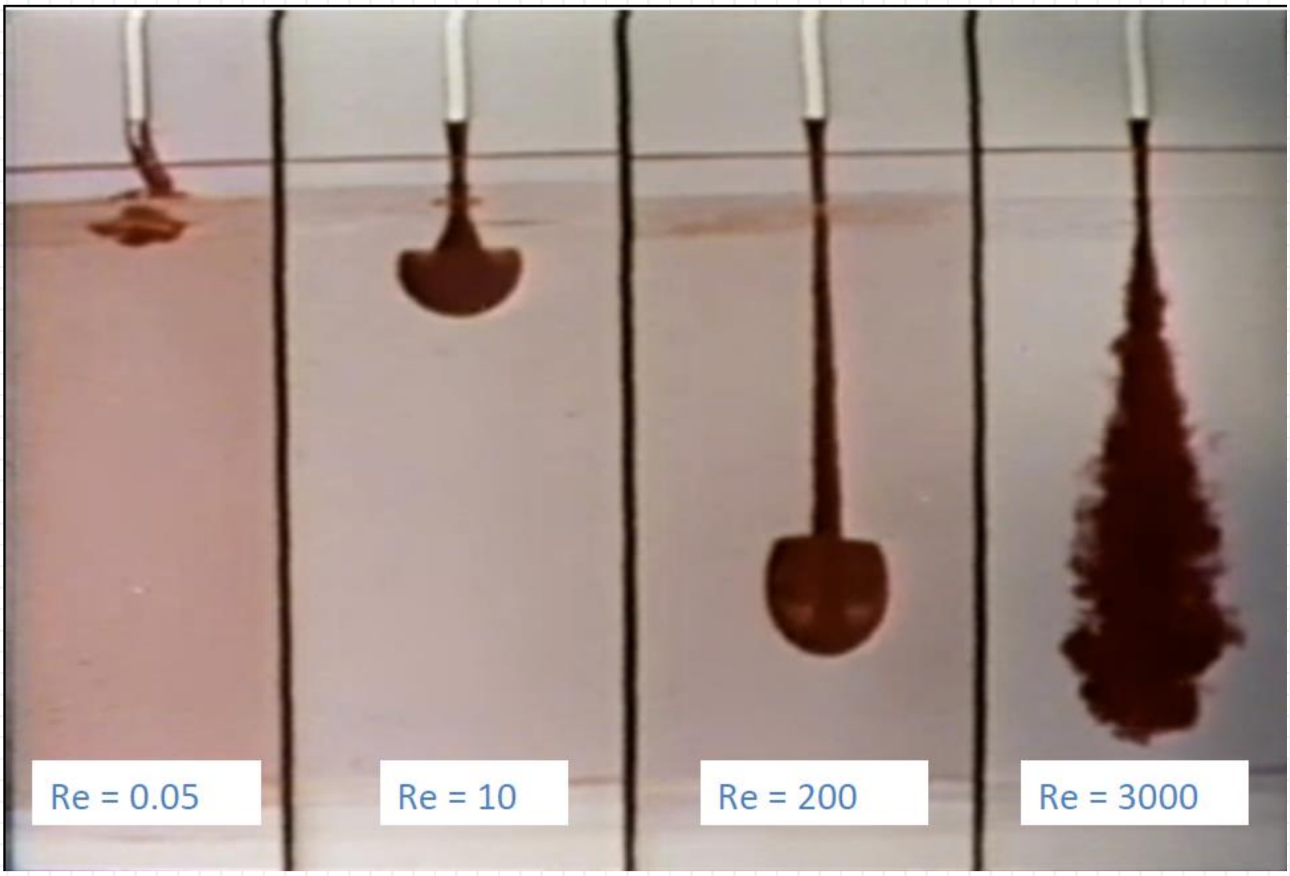


Abbildung 1 National Committee for Fluid Mechanics Films (NCFMF) /Taylor 1964)

(see video)

3.2 Modelling Turbulence

3.2.1 Velocity Statistics

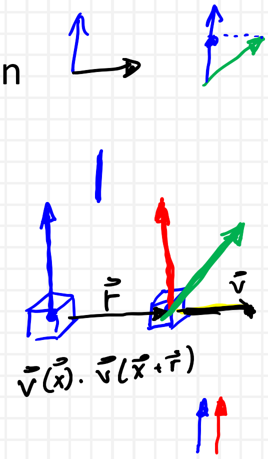
Important property of turbulence: velocity structure, i.e. how does the velocity change from point to point

In turbulent medium, velocity fluctuates in time and space => statistical study

Assumptions:
- homogeneous turbulence (turbulent motions vary only randomly, not systematically)
- isotropic turbulence (no preferred direction)

(neither are true for mol. clouds!)

Autocorrelation function characterizes how $\vec{v}$ varies with position

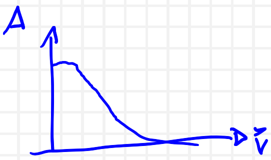


$$A(\vec{r}) = \frac{1}{V}\int \vec{v}(\vec{x})\cdot\vec{v}(\vec{x}+\vec{r})dV \equiv \langle \vec{v}(\vec{x})\cdot\vec{v}(\vec{x}+\vec{r})\rangle$$

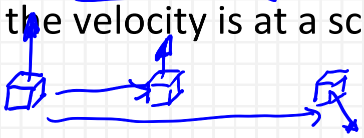
$\langle\dots\rangle$ indicate average over all positions

$A(0) = \langle|\vec{v}|^2\rangle$ RMS velocity in the fluid

Isotropicity: $A(\vec{r})$ cannot depend on direction → $\vec{r} \to |\vec{r}|$



$A(r)$ measures how different the velocity is at a scale r



Alternatively, in Fourier space: the Fourier transform of the velocity field:

$$\tilde{v}(\vec{k}) = \frac{1}{\sqrt{2\pi}}\int \vec{v}(\vec{x})e^{i\vec{k}\cdot\vec{x}}d\vec{x}$$

The the power spectrum is:

$$\Psi(\vec{k}) = \left|\tilde{v}(\vec{k})\right|^2$$

Again, because of isotropicity, the power stepctrum depends only on the magnitude of the wave nu,ber $k = |\vec{k}|$, not its direction.

Instead of Ψ we use the power per unit radius in k-space:

$$P(\vec{k}) = 4\pi k^2 \Psi(k)$$

(Total power integrated over some shell from k to dk in k-space.

Parseval's Theorem

$E_{kin} \propto v^2$

$$\int P(k)dk = \int \left|\tilde{v}(\vec{k})\right|^2 d^3\vec{k} = \int \vec{v}(\vec{x})^2 d^3\vec{x}$$

Integral over power spectral density over all wavenumbers is equal to the integral of the square of the velocity over all space

⇨ if $\rho = const$ the integral of the power spectrum is the kinetic energy per unit mass in the flow!

Wiener-Khinchin Theorem

$P(k)$ is just the Fourier transform of the autocorrelation function

$$\Psi(\vec{\mathrm{k}}) = \frac{1}{(2\pi)^{3/2}} \int A(\vec{r})\, e^{i\vec{\mathrm{k}}\cdot\vec{r}} d\vec{r}$$

- ⇨ Power spectrum at wavenumber k → fraction of total power is in motions at that wavenumber, i.e. on that characteristic length scale ($k = 2\pi/\lambda$).
- ⇨ A power spectrum that peaks at low k → most of the turbulent power is in large-scale motions
- ⇨ A power spectrum that peaks at high k → most of the turbulent power is in small-scale motions
- ⇨ The power spectrum also specifies how the velocity dispersions varies when measured over a region of some characteristic size.

Consider a volume of size l and a velocity dispersion $\sigma_v(l)$ within it. Suppose that the power spectrum is described by a power law $P(k) \propto k^{-n}$. The total kinetic energy per unit within the region is

$$KE \sim \sigma_v(l)^2$$

Above, we saw that we can also write the kinetic energy in terms of the power spectrum integrating over those modes that are small enough to fit within the volume

$$KE \sim \int_{2\pi/l}^{\infty} P(k)dk \propto l^{n-1}$$

normalizing, by defining the sonic scale l_s as the size of a region within which the velocity dispersion is equal to the thermal sound speed of gas, it follows that:

$$\boxed{\sigma_v = c_s \left(\frac{l}{l_s}\right)^{\frac{n-1}{2}}}$$

3.2.2 The Kolmogorov Model and Turbulence Cascades

Basic theory of subsonic, hydrodynamic turbulence: Kolmogorov (1941) (translation: Kolmogorov (1991))

- turbulence occurs when *Re* is large (large range of scales where dissipation is unimportant)
- for incompressible motion transfer of energy can only occur between adjacent wavenumbers
    - energy at a scale *k* cannot transferred directly to some scale $k' \ll k$
    - instead it must cascade through intermediate scales between *k* and *k'*

Energy is injected into a system on some large scale that is dissipationless and it cascades down to smaller scales until at some scale $Re \sim 1$ at which dissipation becomes important.

If turbulence is in equilibrium, the energy at some scale *k* depends only on *k* and on the rate of injection (or dissipation, which are equal) ψ.

Dimensional analysis:

$k \sim 1/L$, power spectrum ~ energy per unit mass ($\sim \text{v}^2 = L^2/T^2$) per unit radius in *k*-space ($\sim 1/L$) $P(k) \sim L^3/T^2$, $\psi \sim L^2/T^3$

$P(k)$ is a function of k and ψ we write (for some dimensionless *C*)

$$P(k) = C\, k^{\alpha} \psi^{\beta}$$

Then

$$\frac{L^3}{T^2} \sim L^{-\alpha} \left(\frac{L^2}{T^3}\right)^{\beta}$$

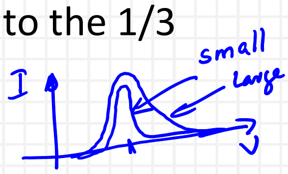
$$L^3 \sim L^{-\alpha+2\beta}$$

$$T^{-2} \sim T^{-3\beta}$$

$$\boxed{\beta = \frac{2}{3}}$$

$$\alpha = 2\beta - 3 = -\boxed{\frac{5}{3}}$$

- power in the flow varies with energy injection rate to the $\boxed{2/3}$
- power distributed such that the power at a given wavenumber k varies as $k^{-5/3}$
  - most power in the largest scale motion
- we find that $\sigma_v \propto l^{1/3}$
  - velocity dispersion increases with size scale as size to the 1/3
  - linewidth-size relation
    - larger regions should have larger linewidths



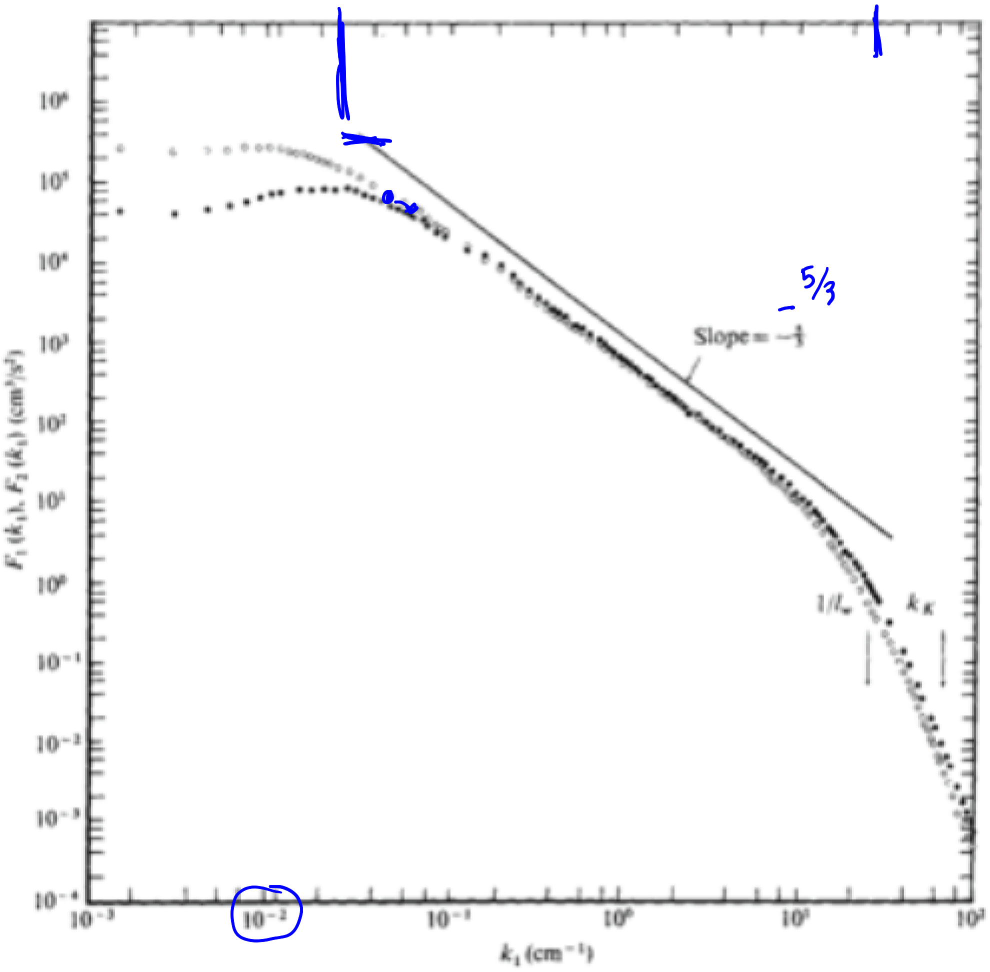


Abbildung 2Power spectrum for turbulence generated in an air jet (Champagne 1978)

3.3 Supersonic Turbulence

3.3.1 Velocity Statistics

real interstellar clouds: $Re \gg 1$, $\mathcal{M} \gg 1$ supersonic

pressure unimportant on scales $L \gg l_s$

viscosity also unimportant on large scales

→ gas tends to move ballistically on large scales

→ on small scales sharp velocity gradients (fast volumes will overtake slow volumes)

→ eventually viscosity will stop the fluid from moving ballistically

⇨ shocks (size scale determined by viscosity)

velocity field: series of step functions
power spectrum: $P(k) \propto k^{-2}$

$k^{-5/3}$

subsonic turbulence: $-5/3$ energy decay via cascade to small scales

supersonic turbulence -2 decay via shock formation

(one single shock generates $P(k) \propto k^{-2}$. i.e. nonlocally couples many scales

⇨ all scales are coupled in shocks

3.3.2 Density Statistics

in subsonic flows, the dominant pressure leads to ~constant density in isothermal gas

in supersonic flows -> highly compressible -> structure of density field

From numerical experiments and empirical arguments:

supersonically turbulent, isothermal medium -> log-normal distribution

$$p(s) = \frac{1}{\sqrt{2\,\pi\,\sigma_s^2}} \exp\left[-\frac{(s-s_0)^2}{2\sigma_s^2}\right], \quad \text{where} \quad s = \ln\left(\frac{\rho}{\bar{\rho}}\right)$$

s: log of density normalized to mean density $\bar{\rho}$, s_0: median of distribution

probability that the density at a randomly chosen point will be such that $\ln(\rho/\bar{\rho})$ is in the range from s to $s + ds$.

Because $\bar{\rho} = \int p(s)\rho ds = e^{s_0+\sigma_s^2/2}$ we find: $s_0 = -\sigma_s^2/2$

probability that a randomly chosen mass element will have a particular density:

consider a volume V, all material with density such that $\ln(\rho/\bar{\rho})$ is in the range from s to $s + ds$. This material occupies a volume $dV = p(s)V$ and has a mass:

$$\begin{aligned} dM &= \rho\, p(s) dV \\ &= \bar{\rho}\, e^s \frac{1}{\sqrt{2\,\pi\,\sigma_s^2}} \exp\left[-\frac{(s-s_0)^2}{2\sigma_s^2}\right] dV \end{aligned}$$

$$p_M(s) = \frac{1}{\sqrt{2\,\pi\,\sigma_s^2}} \exp\left[-\frac{(s+s_0)^2}{2\sigma_s^2}\right] dV$$

The mass PDF is the same as the volume PDF but with the peak shifted from $-s_0$ to $+s_0$:

- ⇨ typical volume element is at a density lower than the mean (because mass is collected into shocks)
- ⇨ typical mass element is in those shocked regions and is at a higher than average density

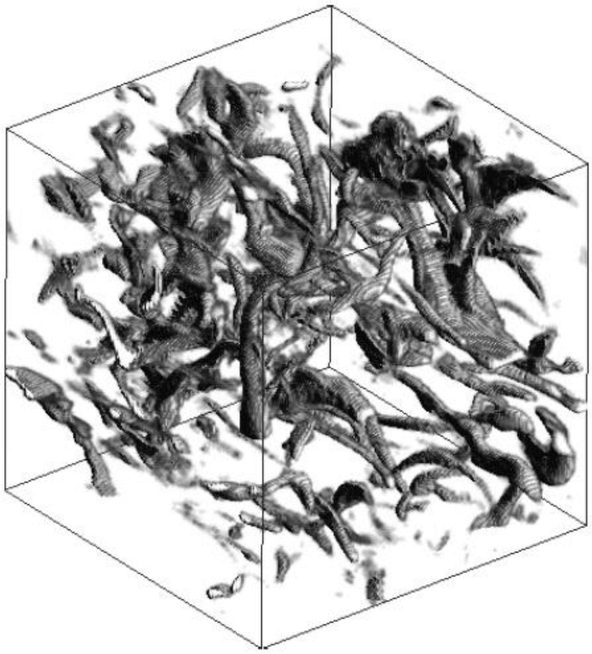


Abbildung 3 Density field of supersonic turbulence (Padoan Z& Nordlund 1999)

Supersonic turbulence:

series of shocks and rarefaction

density multiplied by factors > and < 1

A log-normal process is the statistical realization of the multiplicative product of many independent random variables, each of which is positive.

log-normal in density

dispersion of densities (empirically from num. simulations)

$$\boxed{\sigma_s^2 \approx \ln\left(1 + b^2 \mathcal{M}^2 \frac{\beta_0}{\beta_0 + 1}\right)}$$

b is a number in the 1/3-1 (depends on how compressive vs. selenoidal) the velocity field is

β_0 is the ratio of thermal to magnetic pressure at the mean density and magnetic field strength

4 Magnetic Fields and Magnetized Turbulence